Conformal field theory in the Tomonaga–Luttinger model with the $1/r^{\beta}$ long-range interaction

2006 J. Phys. A: Math. Gen. 39 2161

(http://iopscience.iop.org/0305-4470/39/9/012)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 2161–2175 doi:10.1088/0305-4470/39/9/012

Conformal field theory in the Tomonaga–Luttinger model with the $1/r^{\beta}$ long-range interaction

Hitoshi Inoue[1,2] **and Kiyohide Nomura**[1]

[1] Department of Physics, Kyushu University 33, Hakozaki, Higashi-ku, Fukuoka 812-8581, Japan
[2] West Komoda 3-14-36, Iizuka 820-0017, Fukuoka, Japan

Received 29 September 2005, in final form 18 January 2006
Published 15 February 2006
Online at stacks.iop.org/JPhysA/39/2161

Abstract
We attempt to construct $U(1)$ conformal field theory (CFT) in the Tomonaga–Luttinger (TL) liquid with the $1/r^{\beta}$ long-range interaction (LRI). Treating the long-range forward scattering as a perturbation and applying CFT to it, we derive finite size scalings which depend on the power of the LRI. The obtained finite size scalings give nontrivial behaviours when β is odd and is close to 2. We find consistency between the analytical arguments and numerical results in the finite size scaling of energy.

PACS numbers: 67.20.+k, 05.30.Fk, 05.30.Jp, 05.70.Jk

1. Introduction

Electron systems have attracted much attention in low-energy physics. As the dimension of the electron systems decreases, charge screening effects become less important. In spite of this, models with short-range interaction have been adopted in many researches of one-dimensional electron systems. Recent advanced technology makes it possible to fabricate quasi-one-dimensional systems. Actually at low temperature the effect of Coulomb force has been observed in GaAs quantum wires [1], quasi-one-dimensional conductors [2–4] and 1D carbon nanotubes [5–7].

Systems with the $1/r$ Coulomb repulsive forward scattering were investigated on long distance properties by bosonization techniques [8]. The charge correlation function decays with distance as $\exp(-\text{const}(\ln x)^{1/2})$ more slowly than any power law. The momentum distribution function and the density of state do not show a simple power law singular behaviour. Logarithmic behaviours appear in the power [9]. These mean that the system is driven to the Wigner crystal which is quite different from the ordinary Tomonaga–Luttinger (TL) liquid. The investigation for the interaction $1/r^{1-\epsilon}$ through the path integral approach [10] reconfirms the slower decaying of the single-particle Green function for $\epsilon = 0$ and leads to faster decay for $0 < \epsilon (\ll 1)$ than any power type.

Numerical calculation in the electron system with the Coulomb interaction shows that the larger range of the interaction causes the insulator (charge–density wave) to metal (metallic Wigner crystal) transition [11]. In the spinless Fermion system, convergence of the Luttinger parameters exhibits a quasi-metallic behaviour different from the simple TL one [12].

As we will discuss below, forward scattering is irrelevant for $\beta > 1$. As an instance of the effect of long-range Umklapp scattering, it was reported that the $1/r^2$ interaction changes the system from gapless to gapful through the generalized Kosterlitz–Thouless transition [13].

In this paper we discuss conformal field theory (CFT) in a system with long-range interaction (LRI). The basic assumptions of CFT are symmetries of translation, rotation, scale and special conformal transformation. Besides these, we assume short-range interaction in the CFT. Hence it is a subtle problem whether the CFT can describe a system with LRI.

Concerning LRIs, up to now, solvable models with the $1/r^2$ interaction have been discussed [14–17]. With the Bethe ansatz, the conformal anomaly and the conformal dimensions were calculated and the system proved to be described by the $c = 1$ CFT. In fact the central charge from the specific heat agrees with $c = 1$. On the other hand, the ground-state energy is affected by the LRI and the periodic nature. The effective central charge[3] deviates from $c = 1$.

In general, CFT for LRI which breaks the locality has been left as an unresolved problem. It is significant to clarify the validity of CFT to systems with LRI. We investigate the tight-binding model with the $1/r^\beta$ interaction as one such problem. The low-energy effective model consists of the TL liquid, the long-range forward scattering and the long-range spatially oscillating Umklapp scattering. Extending the arguments appearing in [18] to the TL liquid with the long-range forward scattering, we derive the finite size scalings. In the tight-binding model with the $1/r^\beta$ interaction, we calculate numerically the size dependences of energy and the coefficients of $1/L^y$. And we see numerically the relations between the velocity, susceptibility and Drude weight, which CFT requires.

2. Field theoretical approach

We consider the following tight-binding Hamiltonian of interacting spinless fermions:

$$H = -\sum_{j}^{L} \left(c_j^\dagger c_{j+1} + \text{h.c.}\right) + \frac{g}{2}\sum_{i\neq j}^{L}(\rho_i - 1/2)V(i-j)(\rho_j - 1/2), \tag{1}$$

where the operator c_j $\left(c_j^\dagger\right)$ annihilates (creates) the spinless fermion in the site j and $\rho_j = c_j^\dagger c_j$ is the density operator. In order to treat this model under the periodic boundary condition, we define the chord distance between the sites i and j: $r_{i,j} = \left(\frac{L}{\pi}\sin\frac{\pi(i-j)}{L}\right)$, where L is the site number. Using this, we express the LRI as $V(i-j) = \frac{1}{\left(\frac{L}{\pi}\sin\frac{\pi(|i-j|)}{L}\right)^\beta}$.

By the bosonization technique, we obtain the effective action of the Hamiltonian (1) for the arbitrary filling

$$S = \int \mathrm{d}\tau\,\mathrm{d}x \frac{1}{2\pi K}(\nabla\phi)^2 + g\int \mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}x'\partial_x\phi(x,\tau)V(x-x')\partial_{x'}\phi(x',\tau)$$
$$+ g'\int \mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}x'\cos(2k_F x + \sqrt{2}\phi(x,\tau))V(x-x')\cos(2k_F x' + \sqrt{2}\phi(x',\tau)), \tag{2}$$

where $V(x) = \frac{1}{|x|^\beta}$, K is the TL parameter and k_F is the Fermi wave number. g' is the coupling constant proportional to g. The first term of (2) is the TL liquid and the second term

[3] We define the 'effective central charge' by $c' = \frac{6b}{\pi v}$ in the finite size scaling of the ground-state energy $E_g = aL - \frac{b}{L}$. We use the phrase 'effective central charge' in this sense.

is the long-range forward scattering. The last term is the spatially oscillating Umklapp process which includes $\cos 2\sqrt{2}\phi$, which comes from the interaction between the neighbour sites.

Schulz analysed the effects of the Coulomb forward scattering by the bosonization technique in the electron system [8]. He discussed the quasi-Wigner crystal of electrons due to Coulomb forward scattering. Here we focus on the effects of the $1/r^{\beta}$ forward scattering in spinless fermions system. We treat the action

$$S = \int \mathrm{d}\tau\,\mathrm{d}x \frac{1}{2\pi K}(\nabla\phi)^2 + g\int \mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}x'\partial_x\phi(x,\tau)V(|x-x'|)\partial_{x'}\phi(x',\tau) \tag{3}$$

for any filling k_F. To investigate in the Fourier space, we choose the form $V(x) = \frac{1}{(x^2+\alpha^2)^{\beta/2}}$, where α is the ultraviolet cut-off. In the wave number space, the action (3) is expressed as

$$S = \int \mathrm{d}q\,\mathrm{d}w \left\{\frac{2\pi}{K}(q^2+w^2) + gq^2V(q)\right\}|\phi(q,w)|^2, \tag{4}$$

where $V(q)$ is the Fourier transformation of $V(x)$:

$$V(q) = \frac{2\sqrt{\pi}}{\Gamma(\beta/2)2^{\beta/2-1/2}}(\alpha q)^{\beta/2-1/2}K_{\beta/2-1/2}(\alpha q). \tag{5}$$

Here $K_\nu(x)$ is the modified Bessel function of νth order and $\Gamma(x)$ is the gamma function. From this, the dispersion relation is

$$w^2 = q^2\left\{1 + \frac{gK}{2\pi}V(q)\right\}. \tag{6}$$

The long wavelength behaviours of $V(q)$ are given by

$$V(q) \sim \begin{cases} A + B(q\alpha)^2 + C(q\alpha)^{\beta-1} + \cdots & \beta > 0 \text{ and } \beta \neq \text{ odd} \\ A + B\ln q\alpha + \cdots & \beta = 1 \\ A + B(q\alpha)^2\ln q\alpha + C(q\alpha)^2 + \cdots & \beta = 3 \\ A + B(q\alpha)^2 + C(q\alpha)^4\ln(q\alpha) + D(q\alpha)^4 + \cdots & \beta = 5 \\ \cdots, \end{cases} \tag{7}$$

where A, B, C and D are the functions of β. For the case where $\beta > 0$ and $\neq$ odd, the coefficient $B = B(\beta), C = C(\beta)$ is given by

$$\begin{aligned} B(\beta) &= \frac{\pi^{3/2}}{4}\frac{1}{2^{\beta/2-1/2}\Gamma\left(\frac{5-\beta}{2}\right)\Gamma(\beta/2)\sin\frac{(\beta-1)\pi}{2}} \\ C(\beta) &= -\pi^{3/2}\frac{1}{2^{\beta-1}\Gamma\left(\frac{\beta+1}{2}\right)\Gamma(\beta/2)\sin\frac{(\beta-1)\pi}{2}}. \end{aligned} \tag{8}$$

From equations (6) and (7), we see that the $(q\alpha)^{\beta-1}$ and $\ln q\alpha$ terms for $0 < \beta \leqslant 1$ affect the linear dispersion essentially. In particular, for $\beta = 1$, there is an analysis by Schulz, where the charge–density correlation function is calculated [8]. According to it, in the present spinless case, the LRI drives the ground state from the TL liquid to the quasi-Wigner crystal as $\beta \to 1+$. The slowest decaying part of the density correlation function is given by

$$\langle\rho(x)\rho(0)\rangle \sim \cos(2k_Fx)\exp(-c\sqrt{\log x}), \tag{9}$$

where c is a function of K, which exhibits slower spatial decay than the power decay of the TL liquid.

Then we see the effects of the long-range forward scattering from the standpoint of the renormalization of g. The renormalization group equations of g, v and K are simply derived for long wavelength (see Appendix A). From the renormalization equations, the g terms are relevant for $\beta < 1$, marginal for $\beta = 1$ and irrelevant for $\beta > 1$. Thus, it is expected that the system becomes the quasi-Wigner crystal caused by the forward scattering for $\beta \leqslant 1$ and the system becomes the TL liquid when $\beta > 1$. We see that

$$\Phi(x) \equiv \int dx' \partial_x \phi(x, \tau) V(x - x') \partial_{x'} \phi(x', \tau) \tag{10}$$

has the scaling dimensions $x_g = \beta + 1$ for $1 < \beta < 3$ and 4 for $\beta > 3$. As the weak logarithmic corrections appear for $\beta =$ odd, we distinguish here $\Phi(x)$ for $\beta = 3$ from the scaling functions. We also find the consistency on these scaling dimensions by the CFT. By using the first-order perturbation, we can know the effects of the long-range forward scattering. Based on CFT, the finite size scalings of energies for no perturbations are given [19–22] by

$$\Delta E_n = \frac{2\pi v x_n}{L} \qquad E_g = e_g L - \frac{\pi v c}{6L}, \tag{11}$$

where x_n is the scaling dimension of the primary field denoted by n, v is the sound velocity and c is the central charge. Considering the LRI, we can extract the corrections to these energy size scalings (see the appendices)

$$\begin{aligned} \Delta E_n &= \frac{2\pi v x_n}{L}\left(1 + \frac{g(0)}{x_n}\frac{\text{const}}{L^{\beta-1}} + O(1/L^2)\right) \\ E_g &= (e_g + g(0)\,\text{const})L - \frac{\pi v}{6L}\left(c + g(0)\,\text{const} + g(0)\frac{\text{const}}{L^{\beta-1}} + O(1/L^2)\right), \end{aligned} \tag{12}$$

where $\beta (> 1)$ is not odd. And the constants are the functions of β. Note that for $\beta =$ odd array, the logarithmic corrections appear. They correspond to the integer points of the modified Bessel function, which appear in the long-wave behaviours (7). We can reproduce these anomalies for $\beta =$ odd by the CFT. Moreover from the CFT we can show that there are anomalies in the general excitations and the ground-state energy. The details are shown in the appendices. The $O(1/L^2)$ terms come from the irrelevant field $L_{-2}\bar{L}_{-2}\mathbf{1}$ and the long-range g term. The first equation of (12) means that the long-range forward scattering $\Phi(x)$ has the scaling dimensions $x_g = \beta + 1$ for $1 < \beta < 3$ and 4 for $\beta > 3$, effectively. These respective scaling dimensions are consistent with the estimation from the renormalization group equations of g, which we mentioned above (see the appendices).

The energy finite size scalings (12) mean that the LRI has the higher-order influences than $1/L$ to the excitation energy and the LRI affects the $1/L$ term in the finite size scaling of the ground-state energy. Here we note that it becomes difficult to calculate the central charge from finite size scalings (11) unless the effects of the LRI to the $O(1/L)$ terms are known.

It is notable to compare equations (12) with the case where the perturbations are of short-range type. Ludwig and Cardy calculated the contributions of the short-range perturbation [18]. The results for the irrelevant perturbation, $-g\sum_r \phi(r)$, which has the scaling dimensions $x > 2$, are

$$\begin{aligned} \Delta E_n &= \frac{2\pi v x_n}{L}\left(1 + \frac{g(0)}{x_n} C_{nng}\left(\frac{2\pi}{L}\right)^{x-2}\right) \\ E_g &= (e_g + g(0)\,\text{const})L - \frac{\pi v}{6L}\left(c + g(0)^2\frac{\text{const}}{L^{2x-4}} + O(1/L^{3x-6})\right), \end{aligned} \tag{13}$$

where the $O(g)$ terms do not appear in the ground-state scaling because we set $\langle\phi\rangle = 0$ for the short-range interaction. These results mean that the $x > 2$ irrelevant field has influences of the higher order on the finite size scalings (11). In contrast, we see $\langle\Phi\rangle \neq 0$ in the long-range case, where Φ is defined in equation (10). The LRI has the $O(1/L)$ intrinsic influence on the finite size scaling of the ground-state energy, as appearing in the scalings (12), even if the LRI is irrelevant, that is, $x_g > 2$.

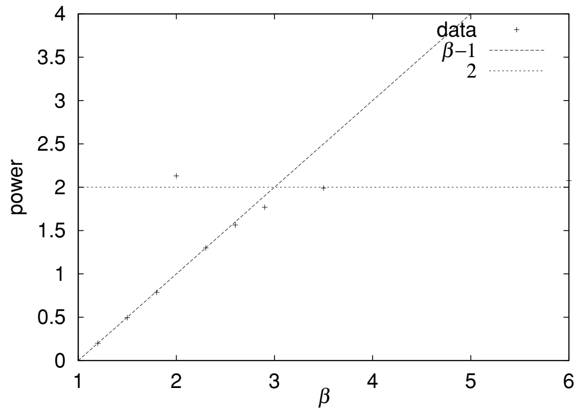

Figure 1. The numerically calculated powers c in the excitation energy $L\Delta E(m = 1/L)$ are shown versus β for $g = 0.5$. Here we use the scaling form $L\Delta E = a + \frac{b}{L^c} + \frac{d}{L^2}$, where a, b, c and d are determined numerically. If the LRI is not present, the energy finite size scaling must take the form $L\Delta E = A + \frac{B}{L^2} + O(1/L^4)$, where A and B are constant values.

3. Numerical calculations

Through the Jordan–Wigner transformation, we transform the model (1) to $S = 1/2$ spin Hamiltonian for the numerical calculations

$$H = -\sum_j \left(S_j^+ S_{j+1}^- + \text{h.c.}\right) + \frac{g}{2} \sum_{i \neq j} S_i^z V(|i-j|) S_j^z. \tag{14}$$

We impose the periodic boundary condition $\mathbf{S}_{L+1} = \mathbf{S}_1$ on this model. Using the Lanczos algorithm we perform the numerical calculations for the Hamiltonian (14).

We have found analytically the corrections to the energy scalings (11) caused by the long-range forward scattering. If the oscillating Umklapp process term of (2) is irrelevant and does not disturb the energy scalings, the finite size corrections due to the forward scattering are expected to appear in the excited state energies and the ground-state energy. We attempt to detect the contribution of the forward scattering.

We numerically calculate the size dependences of the excitation energy $\Delta E(m = 1/L)$ and the ground-state energy $E_g(m = 0)$, $E_g(m = 1/L)$ for $g = 0.5$. Here we define the magnetization $m \equiv \sum_j S_j^z \big/ L$ which is the conserved quantity. Fitting the one-particle excitation energy as $L\Delta E(m = 1/L) = a + \frac{b}{L^c} + \frac{d}{L^2}$, we show the power c versus the powers β in figure 1. We see that the power c agrees with theoretical predictions: $\beta - 1$ except for $\beta = 2$. We shall discuss the $\beta = 2$ case later. Fitting the ground-state energy per site as $E_g/L = a + \frac{b}{L^2} + \frac{c}{L^d}$, we plot the powers d versus β in figure 2. We see that the power d does not show agreements with theoretical predictions $\beta + 1$ in $E_g(m = 0)/L$. These disagreements may be caused by the oscillating Umklapp process which becomes relevant at only $m = 0$ filling. In contrast, the oscillating Umklapp process is irrelevant at $m \neq 0$. Actually, in figure 2, we see that the power d shows agreements with theoretical predictions $\beta + 1$ in $E_g(m = 1/L)/L$ except for $\beta = 2$.

As stated above, for $\beta = 2$, the power c in the excitation energy $L\Delta E(m = 1/L) = a + \frac{b}{L^c} + \frac{d}{L^2}$ apparently shows disagreement with the theoretical value $\beta - 1$ and likewise for $\beta = 2$, the power d in the ground-state energy $E_g(m = 1/L)/L = a + \frac{b}{L^2} + \frac{c}{L^d}$ apparently shows disagreement with the theoretical value $\beta + 1$. We investigate the reason for these

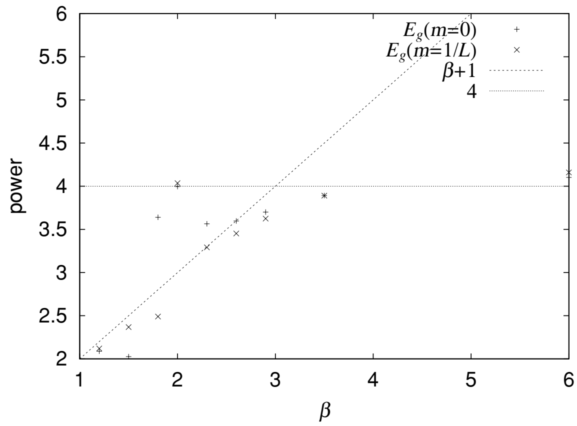

Figure 2. The numerically calculated powers d in the ground-state energies $E_g(m = 1/L)/L$ and $E_g(m = 0)/L$ are shown versus β for $g = 0.5$. Here we use the scaling form $E_g/L = a + \frac{b}{L^2} + \frac{c}{L^d}$, where a, b, c and d are determined numerically. If the LRI is not present, the energy finite size scaling must take the form $E_g/L = A + \frac{B}{L^2} + \frac{C}{L^4}$, where A, B and C are constant values.

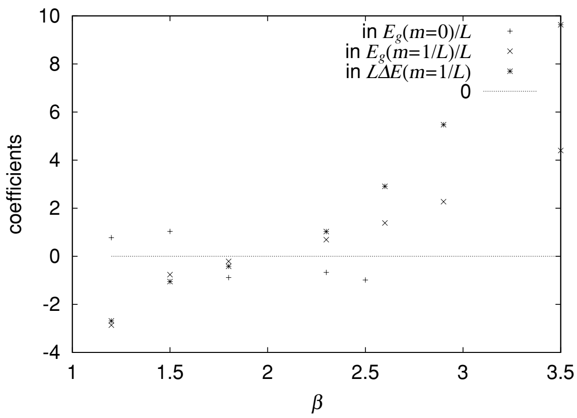

Figure 3. We show the numerically obtained coefficients of $1/L^d$ in the size scalings $E_g(m = 0)/L, E_g(m = 1/L)/L = a + \frac{b}{L^2} + \frac{c}{L^d}$ and the numerically obtained coefficient of $1/L^c$ in the size scaling $L\Delta E(m = 1/L) = a + \frac{b}{L^c} + \frac{d}{L^2}$. We observe that the coefficient of $1/L^d$ in $E_g(m = 1/L)/L$ and the coefficient of $1/L^c$ in $L\Delta E(m = 1/L)$ become small around $\beta = 2$. The coefficients of $1/L^d$ in $E_g(m = 0)/L$ show the different behaviour from that in $E_g(m = 1/L)/L$. This difference may be caused by the spatially oscillating Umklapp process term.

disagreements. In figure 3 we show the numerically obtained coefficient of $1/L^d$ in the size scalings $E_g(m = 0)/L, E_g(m = 1/L)/L = a + \frac{b}{L^2} + \frac{c}{L^d}$ and the numerically obtained coefficient of $1/L^c$ in the size scaling $L\Delta E(m = 1/L) = a + \frac{b}{L^c} + \frac{d}{L^2}$. We observe that the coefficient of $1/L^c$ in $L\Delta E(m = 1/L)$ and the coefficient of $1/L^d$ in $E_g(m = 1/L)/L$ become small around $\beta = 2$. So for $\beta = 2$, the $1/L^2$ dependence appears rather than $1/L$ in $L\Delta E(m = 1/L)$ (see figure 1). Likely for $\beta = 2$, the $1/L^4$ dependence appears rather than $1/L^3$ in $E_g(m = 1/L)/L$ (see figure 2). We observe that the coefficients of $1/L^d$ in $E_g(m = 0)/L$ show the different behaviour from those in $E_g(m = 1/L)/L$ in figure 3. This

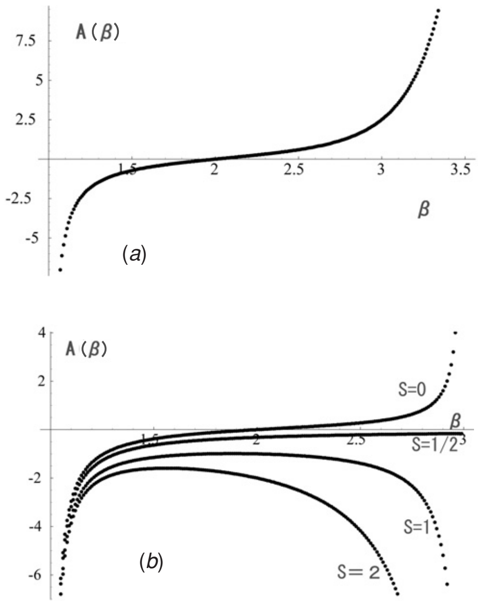

Figure 4. (*a*) $A(\beta)$, the coefficient of $1/L^{\beta}$, in equation (B.7) is shown. We see that $A(\beta)$ has zero point close to $\beta = 2$. This curve coincides with the results from the numerical calculation in the tight-binding model shown in figure 3. (*b*) $A(\beta)$, the coefficient of $1/L^{\beta}$, in equation (B.12) is shown for some s. Analytically only $s = 0$ is meaningful for particle excitations. $A(\beta)$ for $s = 0$ has zero point close to $\beta = 2$. This coincides with the results from the numerical calculation in the tight-binding model shown in figure 3.

difference may come from the spatially oscillating Umklapp process that opens the gap at $m = 0$ and disturbs the finite size scaling.

We can obtain $A(\beta)$ in the scalings (B.7) and (B.12) by evaluating the integrals. The results are shown in figures 4(*a*) and (*b*). The analytical $A(\beta)$ in the scalings (B.7) and $A(\beta, s)$ in the scalings (B.12) for $s = 0$ fit the points in figure 3 well. The curve for only $s = 0$ in figure 4(*b*) shows a good fit. This point shall be discussed later. These reveal that the present numerical calculation of the tight-binding model agrees with the CFT analysis of the long-range forward scattering.

Next, we survey whether the long-range tight-binding model satisfies the necessary condition of the CFT. The operator $\cos\sqrt{2}\phi$ has the scaling dimension $K/2$ and the operator $e^{\pm i\sqrt{2}\theta}$ has $1/2K$ in the regime of the TL liquid. The two quantities $2K/v$ and $vK/2$ are the compressibility and the Drude weight, respectively, in the regime of the TL liquid. If the $c = 1$ CFT is valid for the tight-binding model with the LRI, the two quantities are related to the two excitations with the symmetries $q = \pi, m = 0$ and $q = \pi, m = 1/L$ respectively

$$2K/v = 1/(L\Delta E(m = 1/L, q = \pi)) \equiv \chi \qquad vK/2 = L\Delta E(q = \pi) \equiv D. \tag{15}$$

We show the numerically calculated quantities χ and D in figures 5 and 6, where we use the sizes $L = 16, 18$ and 20 and extrapolate the data. For $g < 0$, χ (which is the susceptibility, irrespective of the CFT arguments) exhibits the rapid increase which suggests the phase separation. In the expression of spin for (1), this phase separation is nothing but the ferromagnetic phase. Hence, for the larger β the point of the phase separation approaches to -1. For $g > 0$ we see the weak tendency that the quantity χ becomes smaller as β is

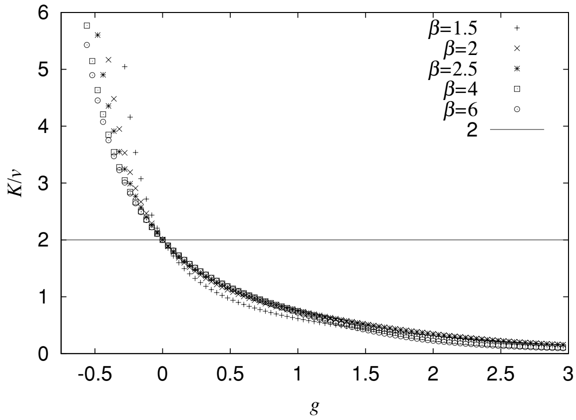

Figure 5. The extrapolated $K/v(= \chi/2)$ is plotted versus the strength g. We use the scaling form $v/K(L) = v/K(\infty) + \frac{a}{L^{\beta-1}}$ for $\beta < 3$, and $v/K(L) = v/K(\infty) + \frac{a}{L^2}$ for $\beta \geqslant 3$, where $v/K(\infty)$, a are determined numerically.

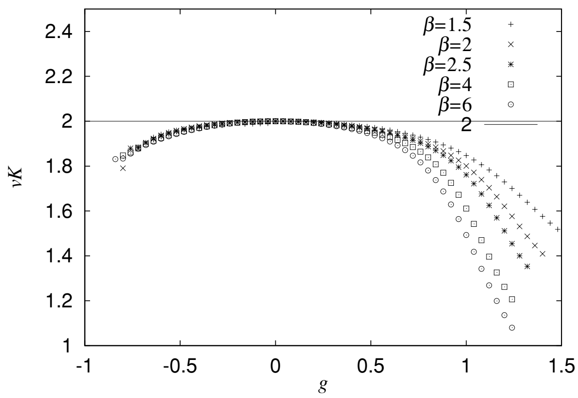

Figure 6. The extrapolated $vK(= 2D)$ is plotted versus the strength g. We use the scaling form $L\Delta E = a + \frac{b}{L^c}$, where a, b and c are determined numerically.

smaller for g less than about 1. We find that the quantity D of $g > 0$ becomes larger as β approaches to $\beta = 1$.

In figure 7 we plot the velocity versus the strength g for the various powers β, where the velocity is defined by

$$v = \frac{L}{2\pi}\Delta E(q = 2\pi/L). \tag{16}$$

We see that the velocities are finite values for $\beta > 1$, as is expected. There are the points where the velocities are zero, implying the phase separation.

In figure 8 we plot the quantity $\frac{D}{\chi v^2}$ versus the strength g for the various powers β. If the present system is described by the $c = 1$ CFT, this quantity is 1 from equations (15). We find the regions where $\frac{D}{\chi v^2} = 1$ in figure 8. The regions become wider as β approaches to 1 for $g > 0$. For larger g, the normalization breaks owing to the generation of mass.

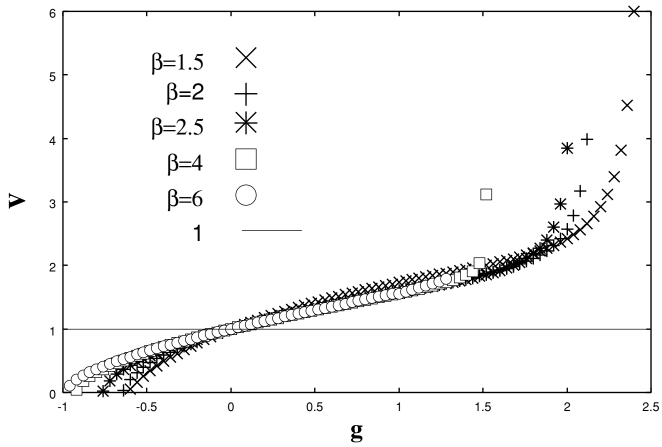

Figure 7. The extrapolated spin wave velocity v is plotted versus the strength g. We use the scaling form $L\Delta E = a + \frac{b}{L^c}$, where a, b and c are determined numerically.

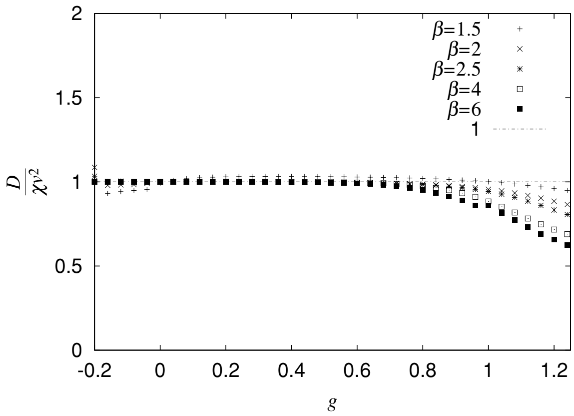

Figure 8. The normalization $\frac{D}{\chi v^2}$ is plotted versus the strength g.

4. Discussions

We have investigated the system with the $1/r^{\beta}$ interaction by applying CFT to it and by the numerical calculation. At first we have analysed the TL liquid with the $1/r^{\beta}$ forward scattering by utilizing the CFT and we have found that the $1/r^{\beta}$ forward scattering works as higher-order corrections in the excitation energy, whereas the effective central charge in the scaling of the ground-state energy depends on the interaction and it deviates from $c = 1$. The deviations are like the solvable $1/r^2$ models [14–17]. Next, we have numerically calculated the ground-state energy and excitations energies in the tight-binding model with the $1/r^{\beta}$ interaction, which is expected to include the above $1/r^{\beta}$ forward scattering in the low energy. The numerical results are in accordance with the analysis with the CFT of the long-range forward scattering. Furthermore, we have numerically checked the normalization $\frac{D}{\chi v^2} = 1$, which is the necessary condition for the $c = 1$ CFT.

For $\beta \approx 2$, the coefficient $A(\beta)$ in the ground-state energy vanishes. This seems to correspond to the exact solution for $\beta = 2$ [17], which states that the finite size scaling of the ground state has no higher-order term than $1/L$. The coefficient $D(\beta)$ of $1/L^3$ in equation (B.9) does not vanish for $\beta = 2$. However, the present argument is the first-order perturbation theory. With higher-order treatments, we may clarify this. In any case, with consistency in many points we could construct CFT in the system with non-local interaction.

The numerical calculations in the tight-binding model support the finite size scalings (B.7) and (B.12). In one-particle excitation energy $L\Delta E(m = 1/L)$, the coefficients of $1/L^\beta$ fit the $s = 0$ case in figure 4. The coefficients from the long-range forward scattering are related to the operator product expansion. We can prove that only the $s = 0$ case is relevant for the particle excitation. Using $\langle\varphi(z)\varphi(z')\rangle = -\frac{K}{4}\ln(z - z')$ and $\langle\bar{\varphi}(\bar{z})\bar{\varphi}(\bar{z}')\rangle = -\frac{K}{4}\ln(\bar{z} - \bar{z}')$, we confirm the operator product expansions

$$\begin{aligned}
&\partial\varphi(z) : \mathrm{e}^{\mathrm{i}\sqrt{2}\theta(z,\bar{z}')} := \frac{-\mathrm{i}\sqrt{2}}{4}\frac{1}{z - z'} : \mathrm{e}^{\mathrm{i}\sqrt{2}\theta(z',\bar{z}')} : + \mathrm{reg} \\
&\bar{\partial}\bar{\varphi}(\bar{z}) : \mathrm{e}^{\mathrm{i}\sqrt{2}\theta(z',\bar{z}')} := \frac{\mathrm{i}\sqrt{2}}{4}\frac{1}{\bar{z} - \bar{z}'} : \mathrm{e}^{\mathrm{i}\sqrt{2}\theta(z',\bar{z}')} : + \mathrm{reg} \\
&T(z) : \mathrm{e}^{\mathrm{i}\sqrt{2}\theta(z',\bar{z}')} := \frac{1}{4K}\frac{1}{(z - z')^2} : \mathrm{e}^{\mathrm{i}\sqrt{2}\theta(z',\bar{z}')} : + \frac{\mathrm{i}\sqrt{2}}{K}\frac{1}{z - z'} : \partial\varphi(z)(: \mathrm{e}^{\mathrm{i}\sqrt{2}\theta(z',\bar{z}')} :) : + \mathrm{reg} \\
&\bar{T}(\bar{z}) : \mathrm{e}^{\mathrm{i}\sqrt{2}\theta(z',\bar{z}')} := \frac{1}{4K}\frac{1}{(\bar{z} - \bar{z}')^2} : \mathrm{e}^{\mathrm{i}\sqrt{2}\theta(z',\bar{z}')} : - \frac{\mathrm{i}\sqrt{2}}{K}\frac{1}{\bar{z} - \bar{z}'} : \bar{\partial}\bar{\varphi}(\bar{z})(: \mathrm{e}^{\mathrm{i}\sqrt{2}\theta(z',\bar{z}')} :) : + \mathrm{reg},
\end{aligned} \tag{17}$$

where we define $T(z) \equiv -\frac{2}{K}(\partial\varphi(z))^2$, $\bar{T}(\bar{z}) \equiv -\frac{2}{K}(\bar{\partial}\bar{\varphi}(\bar{z}))^2$ and $\theta(z, \bar{z}) \equiv \frac{1}{K}(\varphi(z) - \bar{\varphi}(\bar{z}))$. From the first and second equations, we see $C_{\alpha 10} = -\mathrm{i}\sqrt{2}/4$, $C_{\alpha 1\bar{0}} = \mathrm{i}\sqrt{2}/4$ for $\alpha = 1$ and $C_{\alpha 10} = C_{\alpha 1\bar{0}} = 0$ otherwise, where 0 ($\bar{0}$) and 1 denote $\partial\varphi(z)$ ($\bar{\partial}\bar{\varphi}(\bar{z})$) and $: \mathrm{e}^{\mathrm{i}\sqrt{2}\theta(z,\bar{z})} :$. From the third and fourth equations, we see that: $\mathrm{e}^{\mathrm{i}\sqrt{2}\theta(z,\bar{z})}$: have the conformal dimensions $(1/4K, 1/4K)$ and spin 0. As $\mathrm{i}(\partial\varphi(z) - \bar{\partial}\bar{\varphi}(\bar{z}))/2$ is associated with $\partial_\sigma(\sigma)$ for $z = \exp\left(\frac{2\pi w}{L}\right)$, we obtain

$$\langle\alpha|\partial_\sigma(\sigma)|1\rangle = \begin{cases} \frac{2\pi}{L}\,\mathrm{i}(C_{110} - C_{11\bar{0}})/2 = \frac{\sqrt{2}}{4}\frac{2\pi}{L} & \text{for } \alpha = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

which means that only $s = 0$ is relevant for the particle excitation and the last equation in (B.10) has no cosine term.

We discuss the size effects for $\beta = 1$. As seen in equations (A.10) and (B.12), the velocity shows the weak divergence for the size and the Luttinger parameter vanishes gradually for increasing size. This is consistent with the numerical tendency (see figures 8 and 9 in [12]). The size effect of the Drude weight (proportional to the charge stiffness) is now given by $vk/2 \sim$ const as the logarithmic contributions cancel. The numerical data (see figure 7 in [12]) show the metallic behaviour at small and intermediate magnitude interaction strength (larger than the CDW transition point $V = 2$ by the short range interaction). We think that the long-range forward scattering enhances the metallic character. For fairly large interaction the long-range Umklapp scattering becomes relevant and the charge stiffness is suppressed. Finally, we add the size effect of the compressibility

$$L\Delta E(\rho = 1/2 + 1/L) \equiv 1/\chi = O(\ln L) \to \infty, \tag{19}$$

which comes from the results (A.10) by the RG analysis and the CFT arguments. The compressibility χ goes to 0 weakly for increasing size.

To summarize, within the perturbation theory we have constructed the CFT in the TL liquid with the $1/r^\beta$ long-range forward scattering. We have found that the interaction gives a nontrivial behaviour for $\beta =$ odd and $\beta \approx 2$. We have numerically checked the finite size scalings obtained from the CFT in the tight-binding model with the $1/r^\beta$ LRI. Our analysis and numerical calculations exhibit consistency with each other.

Appendix A. Renormalization group equation

$0 < \beta < 1$ *or* $1 < \beta < 3$

We derive the renormalization group equations heuristically. Let us start from the action (4):

$$S = \sum_w \sum_{q=-\Lambda}^{\Lambda} \frac{2\pi}{K}(q^2 + w^2)|\phi(q,w)|^2 + g\sum_w \sum_{q=-\Lambda}^{\Lambda} q^2 V(q)|\phi(q,w)|^2$$
$$= \sum_w \left\{ \sum_{q=-\Lambda/b}^{\Lambda/b} + \sum_{q=-\Lambda}^{-\Lambda/b} + \sum_{q=\Lambda/b}^{\Lambda} \right\} + g\sum_w \left\{ \sum_{q=-\Lambda/b}^{\Lambda/b} + \sum_{q=-\Lambda}^{-\Lambda/b} + \sum_{q=\Lambda/b}^{\Lambda} \right\}. \quad \text{(A.1)}$$

The partition function is

$$Z = \int \mathcal{D}\phi_{\text{slow}}\mathcal{D}\phi_{\text{fast}} \exp\left(-S^0_{\text{slow}} - S^0_{\text{fast}} - S^g_{\text{slow}} - S^g_{\text{fast}}\right). \quad \text{(A.2)}$$

Thus we can integrate S_{fast} ($|q| > \Lambda/b$ component) simply and obtain

$$Z = \int \mathcal{D}\phi_{\text{slow}} \exp\left(-S^0_{\text{slow}} - S^g_{\text{slow}}\right). \quad \text{(A.3)}$$

The remaining procedure of the renormalization is the scale transformation

$$q \to q/b,\ w \to w/b \text{ and } \phi \to \phi b^2, \quad \text{(A.4)}$$

where we choose the dynamical exponent 1. The results are

$$S^0_{\text{slow}} \to S^0$$
$$S^g_{\text{slow}} \to g\sum_w \sum_{q=-\Lambda}^{-\Lambda} q^2 V(q/b)|\phi(q,w)|^2$$
$$\to gb^{1-\beta}\sum_w \sum_{q=-\Lambda}^{-\Lambda} q^2 V(q)|\phi(q,w)|^2, \quad \text{(A.5)}$$

where we use $V(q) - A \sim q^{\beta-1}$ from the behaviours (7). Hence, we obtain renormalization group equation

$$\frac{\mathrm{d}g(b)}{\mathrm{d}b} = (1-\beta)\frac{g(b)}{b}. \quad \text{(A.6)}$$

Substituting $l = \ln b$ into this, we obtain the renormalization group equations

$$\frac{\mathrm{d}g}{\mathrm{d}l} = (1-\beta)g \qquad \frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{v}{K}\right) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{1}{vK}\right) = 0. \quad \text{(A.7)}$$

The TL parameter K is not renormalized but it shifts due to the constant A.

$\beta = 1$

The dispersion relation of the Coulomb interaction includes the marginal part $w \sim q$ and $w \sim q\sqrt{|\ln q|}$ as well as the $\beta > 1$ case. Integrating the fast moving part, we obtain the effective action of the slow part

$$S_{\text{slow}} = \sum_w \sum_{q=-\Lambda/b}^{\Lambda/b} \frac{2\pi}{K}(vq^2 + w^2/v)|\phi(q,w)|^2 + g\sum_w \sum_{q=-\Lambda/b}^{\Lambda/b} q^2 V(q)|\phi(q,w)|^2, \quad \text{(A.8)}$$

where we dare to leave the velocity in the Gaussian part. Note that we do not need the renormalization of the velocity in the case $\beta > 1$. After the scale transformation, we obtain the equations

$$\frac{\mathrm{d}g}{\mathrm{d}l} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{v}{K}\right) = \frac{gA}{2\pi} \qquad \frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{1}{vK}\right) = 0, \tag{A.9}$$

where A is the constant appearing in (7). We see that K and the velocity v are renormalized instead of the no renormalization of g. The forward scattering becomes relevant through K, v and drives the system away from the TL fixed point. Note that this result holds irrespective of any filling k_F. From these equations, the size dependences of v and K are given by

$$v(b) \sim \sqrt{\ln L} \qquad K(b) \sim 1/\sqrt{\ln L}. \tag{A.10}$$

The velocity diverges weakly for long distances, which is consistent with the estimations of $v = \frac{\mathrm{d}w}{\mathrm{d}q}$ from the behaviours (7).

$\beta = 3$

We use $V(q) = A + Bq^2 \ln q + Cq^2 + \cdots$ in the behaviours (7). The g term of (A.1) is

$$\sum_w \sum_{q=-\lambda/b}^{\lambda/b} q^2(g_1(0)q^2 \ln q + g_2(0)q^2), \tag{A.11}$$

where the couplings $g_1(0)$ and $g_2(0)$ are defined by gB and gC, respectively. For the scale transformation (A.4), the g term is changed to

$$\sum_w \sum_{q=-\lambda}^{\lambda} q^2[g_1(0)q^2 \ln q/b^2 + (g_2(0)/b^2 + g_1(0)q^2 \ln q/b^2))]. \tag{A.12}$$

Thus we obtain

$$g_1(b) = \frac{1}{b^2} g_1(0) \qquad g_2(b) = g_1(0)\frac{1}{b^2}\ln\frac{1}{b} + g_2(0)\frac{1}{b^2}. \tag{A.13}$$

By $l = \ln b$, we write this as

$$\frac{\mathrm{d}g_1(l)}{\mathrm{d}l} = -2g_1(l) \qquad \frac{\mathrm{d}g_2(l)}{\mathrm{d}l} = -2g_2(l) - g_1(l). \tag{A.14}$$

$\beta > 3$

This case is same as equation (A.6) putting $\beta = 3$.

Appendix B. CFT in the TL liquid with LRI

The Hamiltonian in the finite strip from the action (3) is

$$H = H_{\mathrm{TL}} + g\int_D \mathrm{d}\sigma_1\, \mathrm{d}\sigma_2 \partial_{\sigma_1}\phi(\sigma_1)\partial_{\sigma_2}\phi(\sigma_2)V(|\sigma_1 - \sigma_2|)\theta(|\sigma_1 - \sigma_2| - \alpha_0), \tag{B.1}$$

where H_{TL} is the TL liquid and D means the region $D = \{|\sigma_1 - \sigma_2| \leqslant L, -L/2 \leqslant \sigma_1, \sigma_2 \leqslant L/2\}$. We introduce the step function $\theta(x)$ to avoid the ultraviolet divergences which come from $V(x)$ and the operator product expansion of $\partial_\sigma\phi(\sigma)$. For the small perturbation g the ground-state energy E_g varies as

$$\begin{aligned}E'_g - E_g &= g\int_D \mathrm{d}\sigma_1\,\mathrm{d}\sigma_2 V(|\sigma_1-\sigma_2|) < 0|\partial_{\sigma_1}\phi(\sigma_1)\partial_{\sigma_2}\phi(\sigma_2)|0 > \theta(|\sigma_1-\sigma_2|-\alpha_0)\\ &= -\frac{g}{4}\int_D \mathrm{d}\sigma_1\,\mathrm{d}\sigma_2 V(|\sigma_1-\sigma_2|)[\langle 0|\partial_{w_1}\varphi(w_1)\partial_{w_2}\varphi(w_2)|0\rangle\\ &\quad+\langle 0|\partial_{\bar w_1}\bar\varphi(\bar w_1)\partial_{\bar w_2}\bar\varphi(\bar w_2)|0\rangle]_{\tau_1=\tau_2=0}\theta(|\sigma_1-\sigma_2|-\alpha_0),\end{aligned} \tag{B.2}$$

where we introduce the coordinates $w=\tau+\mathrm{i}\sigma$ $(-L/2\leqslant\sigma\leqslant L/2, -\infty<\tau<\infty)$ and $|0\rangle$ is the ground state of H_{TL}. From the characters of the Gaussian part (the TL liquid part) we can separate as $\phi(\sigma,\tau)=\varphi(w)+\bar\varphi(\bar w)$ and derive $\langle 0|\partial_{\bar w_1}\bar\varphi(\bar w_1)\partial_{w_2}\varphi(w_2)|0\rangle=0$. The content of the brackets is modified as follows:

$$\begin{aligned}&[\langle 0|\partial_{w_1}\varphi(w_1)\partial_{w_2}\varphi(w_2)|0\rangle+\langle 0|\partial_{\bar w_1}\bar\varphi(\bar w_1)\partial_{\bar w_2}\bar\varphi(\bar w_2)|0\rangle]_{\tau_1=\tau_2=0}\\ &\quad=\frac{K}{4}\left[\left(\frac{2\pi}{L}\right)^{2\Delta}\frac{z_2}{z_1}\frac{1}{\left(1-\frac{z_2}{z_1}\right)^2}+\left(\frac{2\pi}{L}\right)^{2\bar\Delta}\frac{\bar z_2}{\bar z_1}\frac{1}{\left(1-\frac{\bar z_2}{\bar z_1}\right)^2}\right]_{\tau_1=\tau_2=0}\\ &\quad=-\frac{K}{4}\left(\frac{2\pi}{L}\right)^2\frac{1}{2\sin^2\frac{\pi(\sigma_1-\sigma_2)}{L}},\end{aligned} \tag{B.3}$$

where we transform the correlation function $\langle\partial_{z_1}\tilde\varphi(z_1)\partial_{z_2}\tilde\varphi(z_2)\rangle=\frac{K}{4(z_1-z_2)^2}$ in the $\infty\times\infty z$ plane to that in the strip w thorough $z=\exp\frac{2\pi w}{L}$. In the present case $\partial_w\tilde\varphi(w)$ $(\partial_{\bar w}\bar{\tilde\varphi}(\bar w))$ have the spin $s=1(-1)$ and conformal dimension $\Delta=1$ $(\bar\Delta=1)$. Hence we obtain

$$E'_g-E_g=\frac{gK\pi^2}{4}\left(\frac{\pi}{L}\right)^\beta\int_{-1/2}^{1/2}\mathrm{d}x'\frac{1}{(\sin\pi|x'|)^\beta\sin^2\pi x'}\theta\left(|x'|-\frac{\alpha_0}{L}\right), \tag{B.4}$$

where we impose the periodic boundary condition and use the interaction potential $V(x)=1\big/\left(\frac{L}{\pi}\sin\left(\frac{x\pi}{L}\right)\right)^\beta$. Putting $\epsilon=\alpha_0/L$ for convenience, we give the differential of the integral part:

$$\frac{\partial}{\partial\epsilon}\int_{-1/2}^{1/2}\mathrm{d}x'\frac{1}{(\sin\pi|x'|)^\beta\sin^2\pi x'}\theta(|x'|-\epsilon)=-\frac{2}{(\sin\pi|\epsilon|)^\beta\sin^2\pi\epsilon}. \tag{B.5}$$

After integrating the Taylor expansion about ϵ of this quantity, we obtain

$$\begin{aligned}\int_{-1/2}^{1/2}\mathrm{d}x'\frac{1}{(\sin\pi|x'|)^\beta}\frac{1}{\sin^2\pi x'}\theta(|x'|-\epsilon)&=\text{const}+\frac{2}{\pi}\left[\frac{(\pi\epsilon)^{-\beta-1}}{\beta+1}+\frac{\beta+2}{6(\beta-1)}(\pi\epsilon)^{-\beta+1}\right.\\ &\quad\left.+\frac{1}{\beta-3}\left\{\frac{1}{120}(\beta+2)-\frac{1}{72}(\beta+1)(\beta+2)\right\}(\pi\epsilon)^{-\beta+3}+O((\pi\epsilon)^{-\beta+5})\right],\end{aligned} \tag{B.6}$$

where $\beta\neq$ odd. Therefore we can write the corrections in the form

$$E'_g-E_g=\frac{gK}{2}\left[\frac{A(\beta)}{L^\beta}+B(\beta)L+\frac{C(\beta)}{L}+\frac{D(\beta)}{L^3}+O\left(\frac{l}{L^5}\right)\right]. \tag{B.7}$$

Here $B(\beta)$, $C(\beta)$ and $D(\beta)$ are given by

$$\begin{aligned}B(\beta)&=\frac{\alpha_0^{-\beta-1}}{1+\beta}\qquad C(\beta)=\frac{\pi^2(2+\beta)}{6(\beta-1)}\alpha_0^{-\beta+1}\\ D(\beta)&=\frac{\pi^4}{\beta-3}\left\{\frac{1}{120}(\beta+2)-\frac{1}{72}(\beta+1)(\beta+2)\right\}\alpha_0^{3-\beta}.\end{aligned} \tag{B.8}$$

We can obtain $A(\beta)$ by evaluating the above integral numerically. The result is shown in figure 4(a).

For $\beta =$ odd, there exists the logarithmic correction instead of equation (B.7). The results for respective β are

$$E_g' - E_g = \begin{cases} g\left[\frac{A}{L} + BL + \frac{C}{L}\ln\frac{l}{L} + D\frac{l}{L^3} + O\left(\frac{l}{L^5}\right)\right] & \text{for} \quad \beta = 1 \\ g\left[\frac{A}{L^3} + BL + \frac{C}{L} + D\frac{l}{L^3}\ln\frac{l}{L} + O\left(\frac{l}{L^5}\right)\right] & \text{for} \quad \beta = 3 \\ g\left[\frac{A}{L^5} + BL + \frac{C}{L} + D\frac{l}{L^3} + E\frac{l}{L^5}\ln\frac{l}{L} + O\left(\frac{l}{L^7}\right)\right] & \text{for} \quad \beta = 5 \\ \cdots. \end{cases} \tag{B.9}$$

The C terms in equations (B.7) and (B.9) contribute to deviation of the central charge. The present LRI inevitably contains the contribution from the short-range interaction: $\delta(x)$. The C term does not come from the short-range types of interactions because the vacuum-expected value $\langle(\partial_x\phi)^2\rangle$ vanishes. Thus the C term is intrinsic in the present system with the LRI under periodic boundary condition. Because the velocity is not renormalized as we have seen from the renormalization group equations, the C term contributes to the deviations of the effective central charge from the ground-state energy.

Next we derive the corrections for the energy of the excited state

$$\begin{aligned} E_n' - E_n &= g\int_D \mathrm{d}\sigma_1\,\mathrm{d}\sigma_2\, V(|\sigma_1-\sigma_2|)\langle n|\partial_{\sigma_1}\phi(\sigma_1)\partial_{\sigma_2}\phi(\sigma_2)|n\rangle\theta(|\sigma_1-\sigma_2|-\alpha_0) \\ &= g\int_D \mathrm{d}\sigma_1\,\mathrm{d}\sigma_2\, V(|\sigma_1-\sigma_2|)\sum_\alpha\langle n|\partial_{\sigma_1}\phi(\sigma_1)|\alpha\rangle\langle\alpha|\partial_{\sigma_2}\phi(\sigma_2)|n\rangle\theta(|\sigma_1-\sigma_2|-\alpha_0) \\ &= 4g\sum_\alpha C_{nj\alpha}C_{\alpha jn}\frac{(2\pi)^2}{L^\beta}\int_0^{1/2}\mathrm{d}y\frac{1}{(\sin\pi|y|)^\beta}\cos 2\pi(s_n-s_\alpha)y\theta\left(|y|-\frac{\alpha_0}{L}\right), \end{aligned} \tag{B.10}$$

where we use the results by Cardy [22]:

$$\langle n|\phi(\sigma)|\alpha\rangle = C_{nj\alpha}\left(\frac{2\pi}{L}\right)^{x_j}\exp\left(\frac{2\pi\mathrm{i}(s_n-s_\alpha)\sigma}{L}\right). \tag{B.11}$$

Here j means $\partial\phi$ and $|n\rangle$ is the excited state of H_{TL}. We can derive the size dependence of equation (B.10) from likewise treatments as the ground state. After taking the derivative about $1/L$, we expand about $1/L$. Integrating them, we obtain

$$E_n' - E_n = \begin{cases} 16\pi^2 g\sum_\alpha C_{nj\alpha}C_{\alpha jn}\left\{\frac{A(s_n-s_\alpha,\beta)}{L^\beta} + \frac{B(\beta)}{L}\right. \\ \quad \left. + \frac{C(s_n-s_\alpha,\beta)}{L^3} + \frac{D(s_n-s_\alpha,\beta)}{L^5} + O\left(\frac{1}{L^7}\right)\right\} & \beta \neq \text{odd} \\ 16\pi^2 g\sum_\alpha C_{nj\alpha}C_{\alpha jn}\left\{\frac{A(s_n-s_\alpha)}{L} + B\frac{1}{L}\ln\frac{1}{L}\right. \\ \quad \left. + C(s_n-s_\alpha)\frac{1}{L^3} + D(s_n-s_\alpha)\frac{1}{L^5} + O\left(\frac{1}{L^7}\right)\right\} & \beta = 1 \\ 16\pi^2 g\sum_\alpha C_{nj\alpha}C_{\alpha jn}\left\{\frac{A(s_n-s_\alpha)}{L^3} + B\frac{1}{L}\right. \\ \quad \left. + C(s_n-s_\alpha)\frac{1}{L^3}\ln\frac{1}{L} + D(s_n-s_\alpha)\frac{1}{L^5} + O\left(\frac{1}{L^7}\right)\right\} & \beta = 3 \\ 16\pi^2 g\sum_\alpha C_{nj\alpha}C_{\alpha jn}\left\{\frac{A(s_n-s_\alpha)}{L^5} + B\frac{1}{L}\right. \\ \quad \left. + C(s_n-s_\alpha)\frac{1}{L^3} + D(s_n-s_\alpha)\frac{1}{L^5}\ln\frac{1}{L} + O\left(\frac{1}{L^7}\right)\right\} & \beta = 5 \\ \cdots, \end{cases} \tag{B.12}$$

where B are the constant independent of s_n, s_α. Here for $\beta \neq$ odd, $B(\beta)$, $C(s_n - s_\alpha, \beta)$ and $D(s_n - s_\alpha, \beta)$ are given by

$$B(\beta) = \frac{1}{(\alpha_0\pi)^{\beta-1}\pi(\beta-1)}$$
$$C(s_n - s_\alpha, \beta) = \frac{1}{(\alpha_0\pi)^{\beta-3}}\left[\frac{\beta}{6} - 2(s_n - s_\alpha)^2\right]\frac{1}{\pi(\beta-3)} \tag{B.13}$$
$$D(s_n - s_\alpha, \beta) = \frac{1}{(\alpha_0\pi)^{\beta-5}}\left[\left(-\frac{(s_n - s_\alpha)^2}{3} + \frac{1}{180}\right)\beta + \frac{\beta^2}{72} + \frac{2(s_n - s_\alpha)^4}{3}\right]\frac{1}{\pi(\beta-5)}.$$

It is not straightforward to determine $A(s_n - s_\alpha, \beta)$ generally. However, concerning one-particle excitation ($s_n = 0$), we can obtain $A(\beta, s_\alpha)$, which is shown for some s_α in figure 4(*b*). Actually further consideration about the operator product expansion leads to $s_n = s_\alpha = 0$ (see section 4).

We refer to the $O(1/L)$ dependences. These are due to the fact that the LRI includes the short-range-type interaction. Actually we can derive the same form

$$\frac{g}{L}\sum_\alpha C_{nj\alpha}C_{\alpha jn} \tag{B.14}$$

as an ordinary finite scaling by replacing as $V(|x|)\theta(|x| - \alpha_0) \to \delta(x)$. As $(\partial\phi)^2$ is a part of the TL liquid, the $O(1/L)$ term can be removed by subtracting such contributions first. Thus the $O(1/L)$ term is not intrinsic.

Summarizing the discussions in this appendix, we can prove that the Hamiltonian (B.1) is described by the $c = 1$ CFT for $\beta > 1$ in the excitation energy. However, the effective central charge from the ground state depends on the interaction and deviates from 1. We find the nontrivial behaviours when $\beta =$ odd, which corresponds to the integer points of the modified Bessel function as appearing in the behaviours (7).

References

[1] Goni A R *et al* 1991 *Phys. Rev. Lett.* **67** 3298
[2] Darbel B *et al* 1993 *Europhys. Lett.* **24** 687
[3] Nakamura M *et al* 1994 *Phys. Rev.* B **49** 16191
[4] Sekiyama A *et al* 1995 *Phys. Rev.* B **51** 13899
[5] Egger R and Gogolin A O 1997 *Phys. Rev. Lett.* **79** 5082
[6] Kane C, Balents L and Fisher M P A 1997 *Phys. Rev. Lett.* **79** 5086
[7] Yoshioka H and Odintsov A A 1999 *Phys. Rev. Lett.* **82** 374
Yoshioka H and Odintsov A A 1980 *Phys. Rev. Lett.* **82** 3416
Yoshioka H and Odintsov A A 1980 *Phys. Rev. Lett.* **73** 5168
Yoshioka H and Odintsov A A 1980 *Phys. Rev. Lett.* **72** 4009
[8] Schulz H J 1993 *Phys. Rev. Lett.* **20** 1864
[9] Wang D W, Mills A J and Sarma S D 2001 *Phys. Rev.* B **64** 193307
[10] Iucci A and Naon C 1999 *Phys. Rev.* B **61** 15530
[11] Poilblanc D *et al* 1997 *Phys. Rev.* B **56** 456
[12] Capponi S, Poilblanc D and Giamarchi T 2000 *Phys. Rev.* B **61** 13410
[13] Hatsugai Y 1997 *Phys. Rev.* B **56** 12183
[14] Haldane F D M 1991 *Phys. Rev. Lett.* **66** 1529
[15] Sutherland B 1971 *J. Math. Phys.* **12** 246
[16] Romer R A and Sutherland B 1993 *Phys. Rev.* B **48** 6058
[17] Kawakami N and Yang Sung-Kil 1991 *Phys. Rev. Lett.* **67** 2493
[18] Ludwig A W W and Cardy J L 1987 *Nucl. Phys.* B **285** (FS 19) 687
[19] Cardy J L 1984 *J. Phys. A: Math. Gen.* **17** L385
[20] Cardy J L 1986 *J. Phys. A: Math. Gen.* **19** L1093
[21] Cardy J L 1987 *J. Phys. A: Math. Gen.* **19** 5039E
[22] Cardy J L 1986 *Nucl. Phys.* B **270 (FS 16)** 186